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Mach's Principle. Part 1. Initial State of the Universe the Universe

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The integral formulation of equations of general relativity proposed earlier as a mathematical tool for Mach's principle forbids the conventional singular cosmologies but is compatible with the de Sitter initial space.

This paper is stimulated by Raine's fundamental review on Mach's principle (Raine, 1981). Some of the results presented here were published earlier (Altshuler, 1972). The integral form of Einstein's equations as a general covariant selection rule for Machian worlds was proposed by Lynden-Bell (1967) and the author (Altshuler, 1966). This approach was developed by Sciama, Waylen, and Gilman (Sciama *et al.*, 1969; Gilman, 1970). The simplified scalar version of the "integral formulation" was considered also for conformally flat spaces by Maltzev and Markov (1977).

As will be shown in this work the Machian initial conditions preclude cosmologies with a conventional matter equation of state near the singularity—in direct contradiction to Gilman's result (Gilman, 1970). Since the result may depend on the choice of Green's function the whole proof below is carried out in parallel for two integral forms—the SWG one and that of the author. It turns out that our version (but not SWG) permits the de Sitter flat space, i.e., cosmologies with initial $p=-\varepsilon$ equation of state.[2] The possibility of the vacuumlike matter equation of state in a superdense initial epoch was considered earlier by Sakharov (1965) and Gleener (1965). Recently this idea was revived by Starobinsky (1980), Brout *et al.* (1980), and Guth in his "inflationary universe" (1981), and now it draws more attention. (cf. also Linde, 1979; Sher, 1980; Kazanas, 1980; Casher and Englert, 1981; Mukhanov and Chibisov, 1981; Linde, 1982.)

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A few comments on Mach's principle. [For more details and the history of the problem see Raine (1981) and Chin and Hoffmann (1964).] All physical phenomena are obviously noninvariant with respect to general coordinate transformations and, in particular, when accelerated frames are used. Newton attributed this noninvariance to the existence of absolute space, but Mach argued that it is the whole matter of the universe that is responsible for it. Hence, for Mach, the empty space is nonsense (Mach's paradox). Analyzing his general relativity from this point of view Einstein concluded that Machian solutions of the equations

$$R_{ik}-\frac{1}{2}g_{ik}R=\frac{8\pi G}{c^4}T_{ik} \tag{1}$$

(notation is given in the Appendix) are those, where the g_{ik} field is completely determined by the matter distribution in the universe (Einstein, 1955). As was shown in Lynden-Bell (1967) and Altshuler (1966), this Einstein requirement permits a general-covariant rigorous formulation, writing (1) in the form of a nonlinear integral equation

$$g_{ik}(x)=\frac{8\pi G}{c^4}\int G_{ik}^{(\text{zet})\alpha\beta}(x,y|g)\,T_{\alpha\beta}(y)[-g(y)]^{1/2}\,d^4y \tag{2}$$

Here the kernel, retarded Green's function, bitensor in the Riemannian space with the metric g_{ik} (Roman indices are related to point x, Greek to point y), is defined in this space by a linear differential equation:

$$E_{ik}^{mn}G_{mn}^{\alpha\beta}(x,y)=\tfrac{1}{2}(g_i^\alpha g_k^\beta+g_k^\alpha g_i^\beta)\frac{\delta^{(4)}(x-y)}{(-g)^{1/2}} \tag{3}$$

Application of the linear differential operator E_{ik}^{mn} to (2) gives (1) if the following quite weak condition is fulfilled:

$$E_{ik}^{mn}g_{mn}=R_{ik}-\tfrac{1}{2}g_{ik}R \tag{4}$$

The arbitrariness of E_{ik}^{mn} is the main difficulty of the approach. Perhaps the most natural and traditional way is to define E_{ik}^{mn} as the second variational derivative of the gravitational action

$$S=-\int R(-g)^{1/2}\,d^4x \tag{5}$$

In view of condition (4) this differentiation should be done over the mixed components of small metric variations. Let

$$g_{ik}\to g'_{ik}=g_{ik}+h_{ik}=g_{ik}+g_{in}h_k^n$$

and define:

$$E_{i\cdot m\cdot}^{\,k\;\;n} = \frac{1}{(-g)^{1/2}} \frac{\delta^{(2)} S(g'_{00})}{\delta h^i_k \delta h^m_n}\bigg|_{h_{pq}=0} \tag{6}$$

In the Hilbert-deDonder harmonic gauge

$$(h^k_i - \tfrac{1}{2} g^k_i h^n_n)_{ik} = 0 \tag{7}$$

one obtains (Altshuler, 1972; 1968):

$$\begin{aligned} E^{mn}_{ik} = & -\tfrac{1}{2}(g^m_i g^n_k - \tfrac{1}{2} g_{ik} g^{mn})\Box - R_{i\cdot k\cdot}^{\cdot m \cdot n} \\ & + \tfrac{1}{2}(g_{ik} R^{mn} + g^{mn} R_{ik}) - \tfrac{1}{4} g_{ik} g^{mn} R \end{aligned} \tag{8}$$

($i \leftrightarrow k$ symmetrization is implied). Evidently, condition (4) for E^{mn}_{ik} in (8) is valid.

In SWG approach the differential operator is defined by

$$\tfrac{1}{2}(g_{kn}\delta R^n_i + g_{in}\delta R^n_k) = \tilde{E}_{ikmn}\delta g^{mn}$$

and in the gauge (7) it gives (Sciama *et al.*, 1969; Gilman, 1970)

$$\tilde{E}^{mn}_{ik} = \tfrac{1}{4}(g^m_i g^n_k + g^n_i g^m_k)\Box + \tfrac{1}{2}(R_{i\cdot k\cdot}^{\cdot m \cdot n} + R_{i\cdot k\cdot}^{\cdot n \cdot m}) \tag{9}$$

Now, instead of (4), $\tilde{E}^{mn}_{ik}\, g_{mn} = R_{ik}$, the SWG integral formulation differs from (2) and has the form

$$g_{ik}(x) = \frac{8\pi G}{c^4} \int \tilde{G}^{(\text{zet})\alpha\beta}_{ik}(x, y|g_{00})(T_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} T)(-g)^{1/2}\, d^4y \tag{10}$$

where Green's function $\tilde{G}$ is defined by equation (3) with differential operator $\tilde{E}^{mn}_{ik}$.

Equations (2) or (10) preclude all non-Machian empty ($T_{ik} \equiv 0$) and asymptotically empty spaces (Lynden-Bell, 1967; Altshuler, 1966). But the fact that the integral formulation has something to do with the physical Mach's principle is best demonstrated by the following quantitative model example (Altshuler, 1966). Let us consider a thin, hollow, massive cylinder rotating with angular velocity ω around its axis in the empty space [cylindrical analogue of the familiar Lense-Thirring sphere (Thirring and Lense, 1918)]. The rotation of the cylinder induces the rotation (with angular velocity Ω) of the inner inertial frame. Exact solution of Einstein's equations gives for Ω:

$$\Omega = \frac{2a}{1+a}\,\omega, \qquad a \equiv \frac{2\mu G}{c^2}$$

where μ is the cylinder linear mass density. Machian "perfect dragging" $(\Omega = \omega)$ occurs at $a = 1$, i.e., at

$$\mu = \mu_0 = \frac{c^2}{2G} = 7 \times 10^{27} (g/sm)$$

It turns out that the integral forms (2), (10) being rewritten for small metric variations, induced by cylinder rotation, forbid all $\mu \neq \mu_0$. This result depends on the choice of E_{ik}^{mn}. As is shown in Altshuler (1966), a different numerical coefficient before the d'Alembertian in (8) or (9) will cancel the result.

To verify the consistency of Einstein's spaces with (2) we should write down explicitly the initial conditions resulting from (2). Let us put

$$\frac{8\pi G}{c^4} T_{\alpha\beta} = E_{\alpha\beta}^{\gamma\delta} g_{\gamma\delta}$$

[see (1), (4)] into the right-hand side of (2) and integrate it by parts. The validity of (2) demands that the surface integral must be zero and on account of the retarded character of the Green's function we get from (2) the following initial condition:

$$\lim \left\{ \int G_{ik\alpha;\mu}^{\ \ \ \alpha}(x, y) g^{\mu 0}(y) [-g(y)]^{1/2} \, d^3 y \right\} = 0 \tag{11}$$

for

$$y^0 \to -\infty \quad \text{or} \quad y^0 \to 0$$

the initial moment in the singular cosmologies. [We do not concern here difficult problems, arising in spaces with particle horizons (Raine, 1981).]

The general form of the retarded Green's function determined by equation (3) is

$$G_{mn\alpha\beta}(x, y) = \theta(x^0 - y^0) \sum_{\xi} P_m{}^{(\xi)}_n(x) Q_\alpha{}^{(\xi)}_\beta(y) \tag{12}$$

where $\theta(x^0 - y^0)$ is the standard step function; index ξ enumerates all the linear independent solutions (complete set) of the homogeneous linear equations

$$E_{ik}^{mn} P_{mn} = 0 \tag{13a}$$

$$E_{\alpha\beta}^{\gamma\delta} Q_{\gamma\delta} = 0 \tag{13b}$$

(After summation over ξ the right-hand side of (12) is relativistic-invariant). Equations for $P_{mn}(x)$ and $Q_{\alpha\beta}(y)$ coincide since by definition (8) [or (9)] E_{ik}^{mn} is a self-conjugate operator. The Green's function (12) is the solution of (3) if the functions P_{mn}, $Q_{\alpha\beta}$ satisfy two conditions [we write it symbolically without indices; the exact form can be easily reconstructed by substitution of (12) into (3)]

$$\sum_{\xi} P^{(\xi)} Q^{(\xi)}|_{y^0=x^0} = 0$$

$$\sum_{\xi} \frac{\partial P^{(\xi)}}{\partial x^0} Q^{(\xi)}|_{y^0=x^0} \sim \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

The use of (12) reduces (11) to the following set of initial conditions:

$$\lim_{\substack{y^0\to-\infty \\ (y^0\to 0)}} \left[\int Q^{(\xi)\alpha}{}_{\alpha;\mu}(y) g^{\mu 0}(y)[-g(y)]^{1/2}\, d^3y \right] = 0 \tag{14}$$

for all ξ.

Thus to verify (2) we should analyze formulas (13b), (14). Functions $Q_{\alpha\beta}(y)$ depend on the metric g_{ik} as on an external field and (14) selects only those metrics which "screen" all the solutions of (13b). This condition is very strong. Further the Friedman-Robertson-Walker flat model is considered. The space-time metric is

$$ds^2 = c^2\, dt^2 - a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{15}$$

The familiar Einstein equations (the dot denotes differentiation with respect to the time t)

$$3\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{c^4}\varepsilon$$

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(p+\varepsilon) = 0$$

for the matter equation of state $p = \gamma\varepsilon$ (p is the pressure, ε is the energy density, $\gamma = \text{const} \neq -1$) give

$$a(t) = \text{const}\cdot t^m, \qquad m = \frac{2}{3(1+\gamma)} \tag{16}$$

And for the de Sitter flat space ($p = -\varepsilon$) we have $\varepsilon = \varepsilon_0 = \text{const}$,

$$a(t) = \text{const}\cdot e^{\Lambda t}, \qquad \Lambda = \left(\frac{8\pi G}{c^4}\frac{\varepsilon_0}{3}\right)^{1/2} \tag{17}$$

Let us study the initial conditions (14) for the metric (15). The three-space integral in (14) annihilates all $\mathbf{y}$-dependent modes of $Q^{\alpha}_{\alpha}(y^0, \mathbf{y})$. Thus to verify (14) it is sufficient to study only time-dependent solutions of (13b). An explicit calculation of $E_{\alpha}{}^{\gamma\delta}_{\beta} Q_{\gamma\delta}$ by means of (8) for metric (15) shows that the invariant trace Q^α_α of the tensor $Q_{\alpha\beta}$ entangles with its Q^0_0 component. The contraction of equation (13b) and its $\binom{0}{0}$ component are written down here for two functions

$$U_1(t) \equiv Q^0_0 - \tfrac{1}{2}Q^\alpha_\alpha, \qquad U_2(t) \equiv Q^\alpha_\alpha:$$

$$\ddot{U}_1 + \frac{3\dot{a}}{a}\dot{u}_1 + \left(\frac{4\ddot{a}}{a} - \frac{10\dot{a}^2}{a^2}\right)U_1 + \left(\frac{\ddot{a}}{a} - \frac{4\dot{a}^2}{a^2}\right)U_2 = 0 \qquad (18a)$$

$$\ddot{U}_2 + \frac{3\dot{a}}{a}\dot{U}_2 + \left(\frac{2\ddot{a}}{a} + \frac{4\dot{a}^2}{a^2}\right)U_2 - 4\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)U_1 = 0$$

A similar system for the SWG differential operator (9) is

$$\ddot{U}_1 + \frac{1}{2}\ddot{U}_2 + \frac{3\dot{a}}{a}\left(\dot{U}_1 + \frac{1}{2}\dot{U}_2\right) + \left(\frac{2\ddot{a}}{a} - \frac{8\dot{a}^2}{a^2}\right)U_1 - \left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)U_2 = 0$$

$$\ddot{U}_2 + \frac{3\dot{a}}{a}\dot{U}_2 - \left(\frac{4\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2}\right)U_2 - 4\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)U_1 = 0 \qquad (18b)$$

Initial condition (14) now takes the form

$$\lim_{\substack{t\to-\infty \\ (t\to 0)}} [\dot{U}_2^{(\xi)}(t) \cdot a^3(t)] = 0 \qquad \text{for all } \xi \qquad (19)$$

where $\xi = 1, 2, 3, 4$, refers to four linear independent solutions of the systems (18a) or (18b). Equations (18a), (18b), and (19) enable one to study the model (15) for any $a(t)$.

For singular cosmologies (16) the solution of (18a) and (18b) has the form

$$U_1 = C_1 \cdot t^\rho, \qquad U_2 = C_2 \cdot t^\rho \qquad (20)$$

where the power degree ρ acquires four values $\rho(\xi)$, determined by the bi quadratic equation [for the systems (18a), (18b) respectively]:

$$v \equiv \rho^2 + (3m-1)\rho - 6m^2$$

$$v^2 + 6m(2m-1)v - 12m^2(3m-1) = 0 \qquad (21a)$$

$$v^2 - 12m^2 = 0 \qquad (21b)$$

Substitution of (16) and (20) into (19) shows that for this initial condition (when $t \to 0$) to be valid the inequality

$$\rho - 1 + 3m > 0 \qquad (22)$$

is necessary. It is easily seen that (22) cannot be true for all four $\rho^{(\xi)}$ simultaneously in both cases, (21a) and (21b). Thus the cosmologies (16) are inconsistent with the integral forms (2) or (10). We can suppose that Gilman's (1970) opposite result is due to a loss of the most singular modes $Q_{\alpha\beta}(y)$ which determine Green's function behavior near the singularity.

For de Sitter space (17) equations for U_1, U_2 in both systems (18a) and (18b) are separated and to verify (19) it is sufficient to study only the second equations. Substitution of (17) in the second equations in (18a), (18b) gives

$$\ddot{U}_2+3\Lambda\dot{U}_2+6\Lambda^2U_2=0 \tag{23a}$$

$$\ddot{U}_2+3\Lambda\dot{U}_2-6\Lambda^2U_2=0 \tag{23b}$$

In both cases the exponential

$$U_2\sim e^{zt}$$

is the solution of (23a), (23b) for two values of z:

$$Z^{(1,2)}=\tfrac{1}{2}(-3\pm i\sqrt{15})\Lambda \tag{24a}$$

$$Z^{(1,2)}=\tfrac{1}{2}(-3\pm\sqrt{33})\Lambda \tag{24b}$$

The initial condition (19) now takes the form

$$\lim_{t\to-\infty}[Z^{(\xi)}\,e^{(z^{(\xi)}+3\Lambda)t}]=0 \qquad \text{for } \xi=1,2 \tag{25}$$

that is fulfilled for (24a) and not for (24b).

Thus the integral form (2) with Green's function defined by (3), (8) [and not the SWG version (10)] selects the universe beginning its evolution from the de Sitter most symmetric state. This scenario puts the well-known questions: How the Zero-entropy de Sitter state decays into our universe and generates its entropy? And inversely: does the matter go into vacuumlike phase at the finite stage of the collapse and what may be a dynamical mechanism of the entropy and temperature decreasing at such transition? At present, the physical nature of the $p=-\varepsilon$ matter equation of state is understood poorly. From the point of view of Mach's principle this state is singled out because it has no comoving frames (Gleener, 1965) and the very problem of Newton's bucket is eliminated. It must be a strongly bounded collective state with practically infinite excitation energy for any individual particle. But then the question arises: why is the graviton an exception and why is its propagator in (2) supposed to exist in the de Sitter world?

The inequality of gravitational field and all the other fields is a serious drawback of all the "integral formulation" approach to Mach's principle. "Why is the gravitational wave's energy-momentum excluded as the possible source of the metric g_{ik} in the right-hand side of integral form (2)?" This question was put to the author by Professor Wheeler in September 1968[3] and now, 14 years later, I cannot yet answer it. Perhaps, it is natural, since the very notion of Mach's principle needs a better physical foundation. I would like to speculate that a pathway to Mach's principle may be the idea of "confinement" generalized from the Yang–Mills theories to gravity.

Confinement forbids dynamically states with nonzero total charge (color), i.e., with nonzero Gauss flow at infinity. In gravity, unlike gauge vector theories, "charge" (mass) is always of a definite sign, and to make it zero the space should be closed just as Mach's Principle demands. In quantum chromodynamics gluon field exists only in the vicinity of quark sources, inside strings and bags. A similar phenomenon in gravity means probably the absence of absolute space, i.e., Einstein's statement that the g_{ik} field is completely determined by the matter. From this point of view the universe must be considered as a "bag." How should the theory of gravity be modified to include the dynamical confinement, i.e., to be a theory which does not possess non-Machian solutions of field equations?

In the phenomenological description of confinement in vector gauge theories the dielectric permittivity of vacuum ε is supposed to be a dynamical field. In the typical Lagrangian (see, e.g., Kogut and Susskind, 1974; Fukuda, 1978)

$$L = -\frac{1}{4}\,\varepsilon F_{ik}F^{ik} + \frac{1}{2}Z(\varepsilon)\frac{\partial\varepsilon}{\partial x^k}\frac{\partial\varepsilon}{\partial x_k} - V(\varepsilon) \tag{26}$$

the crucial confining role is played by a potential $V(\varepsilon)$ which forbids $\varepsilon = \text{const}$ at infinity, i.e., "pushes" ε to zero in vacuum. The direct gravitational analogy of (26) may be the Brans–Dicke Lagrangian supplemented with a potential (cosmological term) $V(\varphi)$:

$$L = -\varphi R + \omega\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^k}\frac{\partial\varphi}{\partial x_k} - V(\varphi) \tag{27}$$

where $V(\varphi)$ should "push" Brans–Dicke field φ to zero in empty space. In the next paper such a theory with unstable potential $[V(\varphi) \to -\infty$ when $\varphi \to 0]$ is studied in the context of "Big Numbers" problem. A study of Machian consequences of this theory is the scope of a future work.

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APPENDIX: NOTATION

We follow the metric conventions of Landau and Lifshitz (1962). The metric tensor g_{ik} has signature $+---$, and its determinant is denoted by g. The D'Alembertian is defined by $\Box = g^{pq}\nabla_p\nabla_q$, where ∇_p is a covariant derivative; another notation for the covariant derivative is a semicolon. The Riemann–Christoffel tensor is defined in terms of the Cristoffel connection Γ^m_{ik} by

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

The Ricci tensor is $R_{ik} = R^l_{ilk}$.

REFERENCES

Altshuler, B. L. (1966). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **51**, 1143.
Altshuler, B. L. (1968). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **55**, 1311.
Altshuler, B. L. (1972). Abstracts of reports of III Soviet Gravitational Conference, Erevan; thesis, FIAN USSR, 1969.
Brout, R., Englert, F., Frére, J.-M., Gunzig, E., Nardone, P., and Truffin, C. (1980). *Nuc. Phys.*, **B170**, 228.
Casher, A., and Englert, F. (1981). *Phys. Lett.*, **104B**, 117.
Chiu, H., and Hoffmann, W., eds. (1964). *Gravitation and Relativity* W. A. Benjamin, New York.
Einstein, A. (1955) *The Meaning of Relativity* Princeton University Press, Princeton.
Fukuda, R. (1978). *Progr. Theor. Phys.*, **60**, 1152.
Gilman, R. C. (1970). Thesis, Princeton University; *Phys. Rev.*, **D2**, 1400.
Gleener, E. B. (1965). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **49**, 542.
Guth, A. H. (1981). *Phys. Rev.* **D23**, 347.
Kazanas, D. (1980). *Astrophys. J.*, **241**, L59.
Kogut, J., and Susskind, L. (1974). *Phys. Rev.* **D9**, 3501.
Landau, L. D., and Lifshitz, E. M. (1962). *Classical Theory of Fields*, Pergamon, Oxford.
Linde, A. D. (1979). *Rep. Progr. Phys.*, **42**, 389.
Linde, A. D. (1982). *Phys. Lett.*, **B108**, 389.
Lynden-Bell, D. (1967). *Mon. Not. Roy. Astron. Soc.* **135**, 413.
Maltzev, V. K., and Markov, M. A. (1977). Trudi FIAN **96**, 11.
Mukhanov, V. F., and Chibisov, G. V. (1981). *JETP Pisma*, **33**, 549.
Raine, D. J. (1981). *Rep. Prog. Phys.*, **44**, 1151.
Sakharov, A. D. (1965). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **49**, 345.
Sciama, D. W., Waylen, P. C., and Gilman, R. C. (1969). *Phys. Rev.*, **187**, 1762.
Sher, M. A. (1980). *Phys. Rev.*, **D22**, 2989.
Starobinsky, A. A. (1980). *Phys. Lett.*, **91B**, 99.
Thirring, H., and Lense, J. (1918). *Zeitschrift für Physik*, **19**, 156.